White Paper – Hamiltonian for Integer Portfolio Optimisation

Justin Chan

Introduction to Portfolio Optimisation

The portfolio optimisation problem involves choosing proportions of assets in a portfolio such that the portfolio is optimal according to some criteria, represented mathematically by an objective function. Markowitz's (1952) Modern Portfolio Theory modeled this portfolio selection problem through a mean-variance optimisation approach which maximises portfolio returns, as measured by expected returns of assets, and minimises its risk, as measured by covariance matrix of asset returns. The Basic Mean-Variance Model based on the *expected utility maximisation*[1] is given by Equations 1– 3:

$$\underset{w}{\text{argmin}}\left(\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_{ij}w_iw_j - \sum_{i=1}^{N}\mu_i w_i\right) \quad \textbf{(1)}$$

subject to:

$$\sum_{i=1}^{N} w_i = 1 \quad \textbf{(2)}$$

$$0 \leq w_i \leq 1, i = 1, 2, \cdots, N \quad \textbf{(3)}$$

where N is the number of assets considered, w_i is the continuous weights or proportion of asset i $(i = 1, 2, \cdots, N)$ held in the portfolio, σ_{ij} is the returns covariance between assets i and j $(i = 1, 2, \cdots, N; j = 1, 2, \cdots, N)$ and μ_i is the expected return of asset i $(i = 1, 2, \cdots, N)$. Equation 1 maximises the mean-variance utility function subjected to two constraints. Equation 2 is known as the *budget constraint* which ensures all capital budget is invested by requiring the total asset weights to sum to 1. Equation 3 refers to the *long-only constraint* which disallows any short selling of assets where $w < 0$ by ensuring weights for each asset i, w_i do not fall below 0 or exceed 1.

The basic Markowitz model above has two main drawbacks. Firstly, it assumes infinite divisibility of assets as it solves the portfolio optimisation problem in a continuous space by using continuous weights. This makes the optimisation problem convex and its closed-form solution can be easily found by Quadratic Programming. However, the continuous weights allow for unrealistically small trades to move continuously with small shifts in return-forecasts and volatility, attracting significant noise and hence trading costs. Secondly, it assumes the estimated parameters (σ_{ij} and μ_i) encapsulate all risks and returns present in the investment. Hence, the continuous closed-form solution falsely assumes absence of parameters' estimation errors and does not truly reflect real-world uncertainty. Consequently, these two assumptions of the classical model cause it to perform poorly on out-of-sample[2] portfolio risks and returns in a multi-period optimisation as gains from diversification are offset by in-sample estimation errors of the parameters.

[1] This objective function is given by $\frac{\gamma}{2}w^T\Sigma w - \mu w^T$ for a risk aversion coefficient, γ. In our example we chose $\gamma = 2$

[2] While our parameters are estimated using in-sample data, i.e. historical data, the goal of portfolio optimisation is to deliver superior out-of-sample, or forecast, performance

Problem Formulation as an Ising Spin Glass for Quantum Annealing

The first step to incorporating realism into the basic model is to introduce the *round lots*[3] *constraint* to eliminate the assumption of infinite divisibility of assets. This is especially important for institutional investors who are largely limited to trading large blocks of assets in round lots. We therefore assume trades involve lots of assets large enough to comprise integer "portions" of the total capital. This also improves robustness to uncertainty in the out-of-sample optimisation. Instead of allowing our weights to be continuous, we now restrict them to discrete values.

This discrete portfolio optimisation problem has been shown to be NP-complete (Mansini & Speranza, 1999) and fits the Quadratic Unconstrained K-level Optimisation (QUKO) model, a term coined by Hodson (2015) to represent binary encoding of integer variables within the Quadratic Unconstrained Binary Optimisation (QUBO) model. The QUKO model for N QUKO variables is given by Equation 4.

$$H = \sum_{i \in V}^{N} P'_i x'_i + \sum_{\substack{(i,j) \in E \\ i<j}}^{N} Q'_{ij} x'_i x'_j \qquad \textbf{(4)}$$

with each QUKO variable, x' represented as a sum of $K \in \mathbb{Z}^+$ QUBO variables x as given by Equation 5.

$$x'_i = \sum_{k=1}^{K} 2^{k-1} x_{ik} \qquad \textbf{(5)}$$

where K is known as the bit depth. Under this construct, each x'_i has maximum value of $2^K - 1$. The QUKO model can be expressed in terms of QUBO variables, x_i by substituting Equation 5 into Equation 4 which gives us Equation 6.

$$\begin{aligned} H &= \sum_{i \in V}^{N} P'_i \Big(\sum_{k=1}^{K} 2^{k-1} x_{ik}\Big) \\ &\quad + \sum_{\substack{(i,j) \in E \\ i<j}}^{N} Q'_{ij} \Big(\sum_{p=1}^{K} 2^{p-1} x_{ip}\Big)\Big(\sum_{q=1}^{K} 2^{q-1} x_{jq}\Big) \\ &= \sum_{i \in V}^{N} \sum_{k=1}^{K} P'_i 2^{k-1} x_{ik} + \sum_{\substack{(i,j) \in E \\ i<j}}^{N} \sum_{p,q=1}^{K} Q'_{ij} 2^{p+q-2} x_{ip} x_{jq} \end{aligned} \qquad \textbf{(6)}$$

The QUKO expansion into QUBO form can be done automatically by QxBranch's *qxlib* Python package, and further transformed to Ising spin glass form, for execution on a range of solvers including Quantum Annealers (Hodson, 2016).

The integer portfolio optimisation problem is mapped onto the problem Hamiltonian of the Quantum Annealer, H by defining two Hamiltonians: the Optimisation Hamiltonian, H_{opt} which incorporates the expected utility maximisation objective function (Equation 1) and the Penalty Hamiltonian, H_{pen} which incorporates the *budget constraint* (Equation 2) by penalising the solutions that violate the constraint. The *budget constraint* is incorporated by ensuring the QUKO integer weights, x'_i for all assets i sum to the total capital portion, $C \in \mathbb{Z}^+$. We implement an *upper bound constraint* on the asset weights by choosing C such that $2^K - 1 < C < N(2^K - 1)$. This prevents the model from allocating all C capital to one asset and also from investing maximally across all available assets. The nature of QUKO variables being positive integers imposes the *long-only constraint*. The formulation of the problem Hamiltonian, H is given by Equation 7.

[3] Defined as the normal unit of trading of an asset. For example, a round lot for common shares are 100 units

$$H = H_{opt} + H_{pen} = \Big(\sum_{\substack{(i,j)\in E \\ i<j}}^{N} \sigma_{ij} x'_i x'_j - \sum_{i\in V}^{N} \mu_i x'_i \Big) + A\Big(C - \sum_{i=1}^{N} x'_i\Big)^2 \quad (7)$$

with

$$x'_i \in [0, 2^K - 1] \in \mathbb{Z}$$

where A is the penalty coefficient $A > 1$ such that when the solution violates the budget constraint the decrease in H_{opt} is met with greater increase in H_{pen}. The weight for each asset i is given by Equation 8:

$$w_i = \frac{x'_i}{C} \quad (8)$$

Appropriate Bit Depth

The QUKO model stands as the better choice over the QUBO or Ising model in modelling this discrete portfolio optimisation problem as QUKO variables allow for levels of investments while QUBO and Ising variables only allow for binary decisions. What is the appropriate bit depth, K for the portfolio optimisation problem to reflect levels of investment? As $K \longrightarrow \infty$ the asset weights will become continuous, obviating the benefits of discrete optimisation. As $K \longrightarrow 1$, the portfolio will bias toward small number of high risk-adjusted return assets with poor diversity. Selecting an optimal value for K is out of scope for this paper. Some portfolio managers consider a maximum of 10 levels of investment for each asset to be reasonable, $x'_i \in [0, 10]$. This could be approximated by bit depth $K = 4$.

References

Hodson, M. (2015). *A Novel Embedding Technique for Optimization Problems of Fully-Connected Integer Variables. Poster session presented at Adiabatic Quantum Computer Conference 2015, Zurich, Switzerland.*

Hodson, M. (2016). *Rapid Prototyping with Symbolic Computation: Fast Development of Quantum Annealing Solutions*. Unpublished. Washington DC, USA: QxBranch LLC.

Mansini, & Speranza. (1999). Heuristic algorithms for the portfolio selection problem. *European Journal of Operational Research , 114* (2), 219–233.

Markowitz. (1952). Portfolio Selection. *The Journal of Finance , 7* (1), 77–91.

